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ARISTOTLE, ZENO, AND THE STADIUM PARADOX

Kevin Davey

§1: Introduction

Understanding the arguments of the Presocratic philosophers is a difficult business. The task is made even more difficult when the main record of an argument comes from the pen of a critic. Such, of course, is the case with Zeno and his famous paradoxes. We have very few of Zeno's original words; most of what we know comes from Aristotle, whose main purpose in relating Zeno's arguments was to refute them.

This creates a dilemma for the philosopher who wants to get to the bottom of Zeno's paradoxes. Were Zeno's arguments as easily refuted as Aristotle suggested? Or did Aristotle misunderstand Zeno's arguments? To a great degree, one proceeds on faith. Broadly speaking, there are two interpretative possibilities.

The first possibility is that Zeno's arguments were mostly fallacies, and that Aristotle more or less correctly identified these fallacies. This kind of attitude towards Zeno's arguments is found in Barnes:[1]

> Many modern interpreters of Zeno have argued that such and such an account of a paradox is wrong because it attributes such a silly fallacy to a profound mind. Zeno was not profound: he was clever. Some profundities fell from his pen; but so too did some trifling fallacies. And that is what we should expect from an eristic disputant. If we meet a deep argument, we may rejoice; if we are dazzled by a superficial glitter, we are not bound to search for a nugget of philosophical gold.

The second possibility is found in the work of Tannery[2] and Owen.[3] These commentators gave versions of Zeno's arguments that they took to have real persuasive force. A problem with this route, however, is that it often forces one to conclude that Aristotle's criticisms of Zeno miss the point. Zeno is saved, but only by sacrificing Aristotle.

It is difficult to know what sorts of arguments one can give for preferring one of these approaches over the other. Historical evidence is lacking on either side. When Barnes characterizes Zeno as an "eristic disputant," he does so with little historical evidence—in fact, one might argue that Zeno, far from being an eristic disputant, was primarily interested in a careful and rigorous defense of Parmenides, and would not have been taken in by simple fallacies. And yet, there is equally little evidence for the hypothesis that all of Zeno's arguments are *not* correctly represented and refuted by Aristotle. Who then is the hero, and who is the villain—Zeno or Aristotle?

This dilemma is especially visible in the literature on Zeno's fourth paradox, often referred to as "the Stadium." If one sides with Aristotle, the argument is simply an uninteresting fallacy about relative motion, while if one sides with some of the modern interpreters of Zeno, the argument makes a subtle point about atomic theories of space and time that Aristotle completely misunderstood.

There is also a third possibility. When it comes to Zeno's Stadium Paradox, neither Aristotle nor Zeno should be portrayed as victor or villain. Instead, in their discussions of the paradox, both Zeno and Aristotle took reasonable, opposing views on matters concerning the measurement of time that were not properly understood until after the mathematical work of Cantor and Borel in the 19th century. Neither committed any obvious fallacy; and although Aristotle turned out in some sense to be on the right side of the dispute, neither really got to the bottom of things either.

In §2 of the paper, the standard treatments of the Stadium paradox will be discussed, along with their drawbacks. In §3 a new interpretation of the paradox is presented, and its advantages and disadvantages discussed. In §4, Aristotle's reply is examined. In §5 the way in which the work of Cantor and Borel allows us to resolve the dispute between Aristotle and Zeno is discussed, and some concluding remarks made.

§2: The Standard Interpretations

There are two standard, rival interpretations of Zeno's Stadium argument.[4]

The primary surviving text we have for Zeno's Stadium argument is from Book VI of Aristotle's Physics (239b33):

> The fourth argument is that concerning equal bodies which move alongside equal bodies in the stadium from opposite directions—the ones from the end of the stadium, the others from the middle—at equal speeds, in which he thinks it follows that half the time is equal to its double.

It is from this text that both interpretations construct the details of Zeno's argument.

§2.1: The First Interpretation

Begin with three rows of bodies (the A's, B's, and C's), as in Figure 1. Four objects in each row are shown, though the exact number is unspecified, and presumably irrelevant.

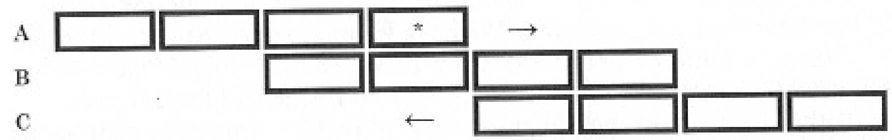


Figure 1: Initial position

Imagine that the A's move to the right, the B's stay stationary, and the C's move to the left, until the arrangement in Figure 2 is produced:

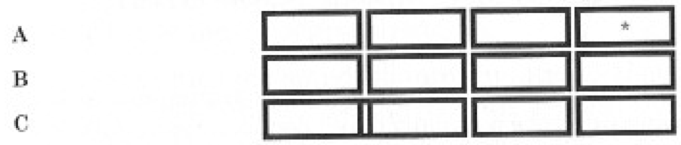


Figure 2: Final position

Consider the rightmost A (labeled with an asterisk (*) in both Figures 1 and 2.) It moves alongside[5] two of the B's in the transition from Figure 1 to Figure 2. However, it also moves alongside four C's in the transition from Figure 1 to Figure 2. If we assume that the amount of time it takes an A to move alongside a B is the same as the amount of time it takes an A to move alongside a C, and that this time is equal to t, we then have $2t=4t$; i.e., $t/2=t$. This is Zeno's paradoxical conclusion. (It is generally accepted that the Greek phrase translated as "half the time is equal to its double" could indicate either an equation of the form $t/2=2t$ or $t/2=t$ depending on what one takes the referent of "it" to be; most interpreters opt for the latter.)

Aristotle's reply to this conclusion is the obvious one—the amount of time it takes an A to move alongside a B is *not* the same as the amount of time it takes an A to move alongside a C, and so Zeno's conclusion is unjustified. See Physics, book VI, 239b33:

> The fallacy consists in requiring that a body traveling at an equal speed travels for an equal time past a moving body and a body of the same size at rest. That is false.

According to this interpretation of the Stadium argument, Zeno's argument involves a very basic error. In fact, Zeno's error is so egregious that one might begin to doubt the accuracy of the interpretation. In response to this sort of concern, defenders of this interpretation of the Stadium paradox have argued that although the error that Aristotle points out seems a triviality to us, it was far from a triviality to the ancient Greeks. For instance, Sorabji argues:[6]

> Now, in Zeno's time, fallacies about the relativity of motion would not have been easy to detect. We have Plato's explicit testimony that Zeno was confused about relativity. (Prm. 128E–130A) He supposed it would be paradoxical if something could be both like and unlike, both one and many, both in motion and at rest. . . . One thing that had been generated by Zeno's predecessor Heraclitus was still worrying Aristotle in the following century: the road from Athens to Thebes may be uphill, while the road from Thebes to Athens is downhill. . . . Viewed in a historical context, then, there is nothing surprising in the mistake about relativity which Aristotle ascribes to Zeno.

And Booth argues:[7]

> It seems to me that those who try to make Zeno's arguments better than they probably were, are not really doing Zeno a service; they are merely showing a gross lack of imagination in regard to the limitations of Zeno's time. They fail to realize that in these early times, such clear formulations as "Distance equals Speed multiplied by Time" had not been made. If they could realize that Zeno's examples of Achilles and the Stadium were perhaps the first inklings that man ever had of such simple equations, they would arrive at a far higher estimation of Zeno's true greatness. In order to praise Zeno, there is no need to slur over the evident shallowness of the paradoxes as posed; but there is every need to understand him in relation to his own times.

There is something unpersuasive about these defenses of Zeno. In the passages of the *Parmenides* to which Sorabji refers, the subject of motion is not even mentioned—the discussion is mainly about the forms of "like" and "unlike," and it seems difficult to infer anything about the depth of Zeno's understanding of motion from this. It is also unclear what the connection is supposed to be between puzzles about roads being both uphill and downhill and Zeno's purported belief that the time it takes an A to move alongside a B is the same as the time it takes an A to move alongside a C. Is the thought here that just as it is puzzling to say that a road can be both uphill and downhill, so too would it be puzzling to say that the time it takes an A to move alongside a B is different from the time it takes an A to move alongside a C? How exactly is this supposed to follow? Would it also be puzzling to say that Alice is shorter than Bob, but not shorter than Charlie? These sorts of maneuvers surely make Zeno look worse, not better.

Booth exaggerates when he suggests that, without access to formulae like $v=d/t$, errors of the kind that Aristotle takes Zeno to commit are inevitable. To recognize that the amount of time it takes an object A to move alongside another object B might depend on the speed of B does not require any sophisticated mathematical machinery; it does not require any mathematical formula such as $v=d/t$, and it does not even require a well developed concept of velocity. Indeed, it is difficult to imagine that someone who recognized the rather subtle point that Achilles must do infinitely many things in order to catch the tortoise (getting to where the tortoise started, getting to where the tortoise has subsequently moved, and so on), might nevertheless have thought that the amount of time it takes an object A to move alongside another object B is independent of the type of motion that B undergoes. So in spite of Sorabji's and Booth's insistences to the contrary, a problem with the standard interpretation of Zeno's Stadium paradox is that it makes Zeno look too foolish.

§2.2: The Second Interpretation

Dissatisfied with the standard interpretation, other interpreters have tried to reconstruct the Stadium paradox in a way that makes Zeno look better. This work includes that of Tannery and Owen.[8]

According to this rival interpretation, the main goal of Zeno's Stadium argument is a refutation of an *atomic* theory of space and time. By an atomic theory of space and time, what is meant is a theory according to which:

(Ai) Time can be divided into *instants* of equal but non-zero duration, within which no change occurs;

(Aii) Any bounded stretch of time contains at most finitely many such instants;

(Aiii) Any line can be divided into *points* of equal but non-zero magnitude, which are themselves indivisible; and

(Aiv) Any bounded segment of a line contains at most finitely many such points.

Furthermore, an *orthodox theory of motion* is defined to be a theory of motion that satisfies the following:

(Bi) If a moving object X passes an object Y of the same width, then at some instant of time, objects X and Y must be lined up,

(Bii) If an object X is in motion, then at distinct times it occupies distinct locations.

Suppose that each of the A's, B's, and C's occupy a single atomic point of the lines on which they are located. Assume also that there is no space between the consecutive A's, consecutive B's, or consecutive C's. (It is generally also assumed that the A's move to the right at a rate of one atomic point per atomic instant of time, and that the C's therefore move to the left at the same rate, even though we will see that this assumption is unnecessary.) Finally, assume an orthodox theory of motion.

Now, consider again the rightmost A (labeled with an asterisk in Figures 1 and 2.) How much time *t* does it take for this object to reach its final destination, in which it is lined up with the rightmost B and the rightmost C?

Well, the rightmost A must, at some stage, line up with the leftmost C, by (Bi). In fact, this must happen after only one atomic unit of time. To see why, note that after one atomic unit of time, both the rightmost A and the leftmost C must have moved (by Bii). Where could they have moved? The rightmost A cannot have overtaken the leftmost C, because otherwise there will be no instant at which these two objects line up, in violation of (Bi). But the rightmost A cannot be "partially" lined up with the leftmost C, as in Figure 3, or else points on a line would be divisible, contradicting (Aiii).

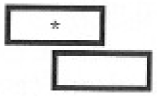


Figure 3

Therefore, after one atomic unit of time, the rightmost A must be lined up with the leftmost C. By a similar argument, after *another* atomic unit of time, the rightmost A must be lined up with the next to leftmost C, and after another atomic unit of time, the rightmost A must be lined up with the third to leftmost C, and so on. The number *t* of atomic units of time that it takes for the rightmost A to get to its final destination is therefore equal to the number of C's.

But now, instead of asking how long it takes the rightmost A to overtake all the C's, ask how long it will take the rightmost A to overtake the B's that it has yet to overtake when in the initial position. And, by exactly the same reasoning, the number *t* of atomic units of time that it takes for the rightmost A to get to its final destination is equal to the number of B's it has yet to overtake when in the initial position.

But the number of B's that the rightmost A has yet to overtake when in the initial position is just half the number of C's. And therefore $t/2=t$, as before.

This interpretation of the Stadium has the advantage that it does not rest on a simple fallacy. But there are still worries about this interpretation that are worth voicing.

One objection is that there is no mention of "atomic" distances or units of time in Aristotle's text. But this is not fatal, as Aristotle may have simply assumed that the reader was familiar with those details of the argument, or may have thought that those details of the argument were not relevant to its refutation.

A second objection is that it is not clear that at Zeno's time there *were* any atomists, and thus it is not clear that the Stadium paradox so interpreted had any real target. But such an objection is not fatal, for even if there were no atomists at the time, the *possibility* of an atomic theory is something Zeno may have been aware of, and felt necessary to refute.

A third objection is that one might think there are simpler arguments against the conjunction of an atomic theory of space and time, and an orthodox theory of motion. In particular, under such assumptions, it seems to follow that all moving objects travel at the same rate of one atomic unit of space per atomic unit of time. For objects cannot "pass over" points of space, lest (Bi) be violated, and moving objects must be in distinct parts of space at distinct times, lest (Bii) be violated. It follows that all moving objects must move at a rate of one atomic unit of space per atomic unit of time. But moving objects can surely have different speeds. Therefore, the conjunction of the atomic theory of space and time with an orthodox theory of motion cannot be correct.[9] But again, this objection does not seem to be fatal. For even if all moving objects move with the same speed, the Stadium paradox shows that there are *still* problems, if objects can move in different directions. The extra complexity is therefore put to use, and cannot be counted against the argument.

A fourth objection is that, insofar as the Stadium argument is supposed to prove the impossibility of an atomic theory of space and time, it is a *bad* argument. In particular, the argument relies critically on an orthodox theory of motion, and so one might reasonably conclude that all it proves is that an atomic theory of space and time must have an *unorthodox* theory of motion attached to it. But again, this objection is hardly fatal. If we are prepared to go with the thought that the Stadium argument is directed against an atomic view of space and time (even though Aristotle does not directly say so) then why not go one step further and say that the Stadium argument is directed against the *conjunction* of an atomic view of space and time with an orthodox theory of motion (even though Aristotle does not directly say so)?

Against this possibility, one might argue that there is reason to think that Zeno's argument is directed *solely* against the atomic theory. Owen[10] maintains, for instance, that Zeno's other paradoxes (including the paradoxes of plurality) can be naturally read as refuting alternative (i.e., non-atomic) theories of space and time, leaving the Stadium paradox to refute atomism. But considerations of this sort are inconclusive. Zeno is believed to have composed at least forty paradoxes,[11] and so it would be a striking coincidence if the handful that have survived formed a "perfect set" that exhaustively refute all possible theories of space, time and motion. If Zeno's paradoxes form some sort of system, and we are committed to making Zeno look smart, then why not acknowledge the possibility that Zeno refuted non-orthodox theories of motion elsewhere, and that Zeno is therefore entitled to assume an orthodox theory of motion when he deals with atomic theories of space and time in the Stadium argument?

None of the above objections are fatal. But a more serious problem with this interpretation of the Stadium paradox is that, if it is right, Aristotle's reply misses the mark very badly. Certainly, Aristotle is correct to point out that the amount of time it takes an A to move alongside a B is *not* the same as the amount of time it takes an A to move alongside a C. But, of course, *Zeno agrees* with this—the whole point of the Stadium argument is to show that an atomic theory of space and time, coupled with an orthodox theory of motion, leads, in certain cases, to the negation of Aristotle's claim—which is *precisely why* the Stadium argument is a *refutation* of an atomic theory of space and time conjoined with an orthodox theory of motion.

Imagine a reductio ad absurdum in which we assume X, derive both Y and ~Y, and so infer ~X. Suppose that Y is false. If someone said that because Y has been derived, and Y is false, something must be wrong with this derivation of ~X, we would send this person to an introductory logic class. Yet if this interpretation of the Stadium argument is right, then this is precisely the error that Aristotle makes. So a serious problem with this interpretation of the Stadium paradox is that it makes Aristotle too foolish.

The standard reading of the Stadium paradox that takes Aristotle's criticism at face value makes Zeno look too foolish. Insofar as we try to save Zeno, we do so by making Aristotle look too foolish.

§3: A New Interpretation

Assume, as before, that the A's, B's, and C's occupy a single, indivisible point of the lines on which they are located, and that there is no space between the A's, between the B's, or between the C's. To say that there is no space between the A's is *not*, of course, to presuppose that, for each A, there is another A immediately to its left or right. Rather, it is just to say that *if* there are points of space between any (not necessarily consecutive) pair of A's, then all such points of space are occupied by other A's.

This really amounts to thinking of the A's as a single body, composed of indivisible parts. The A's are then just the indivisible parts of the solid body that they compose. The same, of course, goes for the B's and C's. Thus, we are to imagine 3 solid objects of equal size that start in the positions depicted in Figure 4:

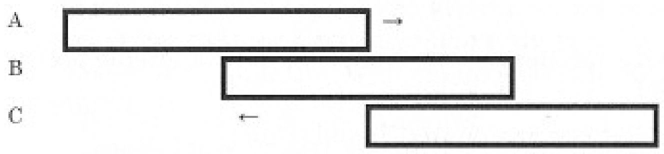


Figure 4

With an exception noted in the next paragraph, no assumptions are made about the nature of the indivisible parts of which these extended bodies are composed. Do they themselves have non-zero magnitude? Are they infinite in number? For each indivisible part, is there another indivisible part to its immediate left or right? Answers to these questions are left unspecified, and thus, an atomic conception of space and time is not presupposed. It is only assumed that space, time, and matter are divisible into indivisible parts.

The only substantive assumption about indivisible parts is that if two objects O_1 and O_2 have the same length, then the indivisible parts composing O_1 may be put in 1-1 correspondence with the indivisible parts composing O_2. (The converse is obviously not assumed.) It would actually suffice to use a principle weaker than this, but the principle just stated is likely to be accepted by many atomists and non-atomists, without presupposing anything like an atomic theory of space and time, and so is reasonable to assume.

It is also assumed that *time* can be divided into indivisible instants, although again, we make no assumptions about the nature of these instants.

As usual, imagine that the A's move to the right, and the C's move to the left with the same speed, but opposite direction, until they are all lined up as in Figure 5:

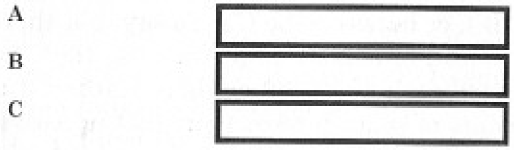


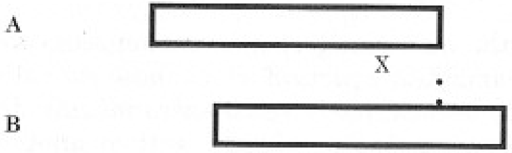
Figure 5

Now consider the very rightmost point of A—call it X. Assume that at any given moment between the start and the end of the motion, X will line up with some point in the right half of B. Conversely, any point in the right half of B will line up with X at some time during the motion. Let us make these assumptions explicit:

(Ci) Any point in the right half of B lines up with X at a unique time during the motion.

(Cii) At all times during the motion, X lines up with a unique point in the right half of B.

Figure 6 depicts what is going on in (Ci) and (Cii):



X only lines up with points from the *right half* of B.

Figure 6

Now, (Ci) just follows from the fact that an orthodox theory of motion is assumed.

For, in the course of the motion, X passes every point in the right half of B. Therefore, if (Bi) is correct, for each point in the right half of B, there is some moment of time at which X and that point are lined up. Moreover, this time must be unique. To see why, fix some point Y in the right half of B, and assume that X lines up with Y at distinct times t_1 and t_2, with t_1 prior to t_2. Then X must also line up with Y at all times t between t_1 and t_2—for if X lined up with Y at t_1, and if at some time after t_1, and prior to t_2, X did *not* line up with Y, then at such a time X must have moved to the right of Y—making it impossible for X to line up with Y again at t_2. But if X is lined up with Y at all times t between and including t_1 and t_2, then this contradicts the second clause (Bii) of our orthodox theory of motion, according to which objects in motion occupy distinct locations at distinct times.

So (Ci) is a consequence of our orthodox theory of motion. What about (Cii)? Assume that X lines up with none of the B's. This would entail that there was space between the B's, which we have explicitly assumed not to be the case. Furthermore, it is obvious that X cannot line up with *more* than one point in the right half of B at any instant of time. Thus (Cii) follows.

It follows from (Ci) and (Cii) that the moments of time that pass in the motion may be put in 1-1 correspondence with the points in the right half of B, i.e.:

(Ciii) The number of instants of time that pass in the motion is equal to the number of points in the right half of B.

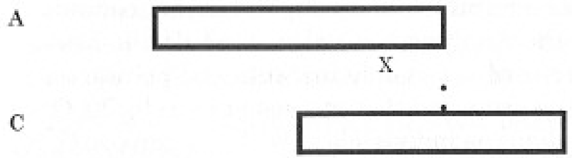
When in (Ciii) it is said that two sets have the same *number*, we simply mean that they may be put in 1-1 correspondence. (Thus, our notion of number is just the modern mathematical notion of "cardinality.")

But similarly, note that at any given moment between the start and the end of the motion, X will also line up with some unique point in C. Conversely, any point in C will line up with X at some time during the motion. So:

(Civ) Any point in C will line up with X at some unique time during the motion.

(Cv) At all times during the motion, X lines up with some unique point in C.

Figure 7 depicts what is going on in (Civ) and (Cv):



X can line up with *any* point from C.

Figure 7

In analogy with (Ci), (Cii), and (Ciii), it follows from (Civ) and (Cv) that:

(Cvi) The number of instants of time that pass in the motion is equal to the number of points in C.

However,

(Cvii) The number of points in C is twice the number of points in the right half of B.

By this, what is meant is that the set of points in C may be placed in 1-1 correspondence with *two* copies of the set of points in the right half of B. That this is so follows from the principle that if two objects O_1 and O_2 have the same length, then the indivisible parts that compose O_1 may be put in 1-1 correspondence with the indivisible parts that compose O_2.

In more detail, the argument is as follows: the set of points in C may be placed in 1-1 correspondence with the set of points in B, by the principle just cited. Therefore, the set of points in C may be placed in 1-1 correspondence with the union of the set of points in the left half of B, and the set of points in the right half of B. However, from the same principle, the set of points in the left half of B may be placed into 1-1 correspondence with the set of points in the right half of B. Therefore, the set of points in C may be placed in 1-1 correspondence with *two* copies of the set of points in the right half of B.

From (Ciii), (Cvi) and (Cvii), we finally have:

Conclusion: The number of instants of time that pass during the motion is equal to two times the number of instants of time that pass during the motion.

As before, when it is said that the number of instants of time that pass during the motion is equal to two times the number of instants of time that pass during the motion, what is meant is that the set of instants of time that pass during the motion may be put in 1-1 correspondence with two copies of the set of instants of time that pass during the motion. Thus, the usual problematic conclusion is obtained.

This interpretation has a number of virtues. It does not commit any obvious fallacy. In fact, the argument is *valid*—and if it is assumed that time and space consists of indivisible instants and points, and an orthodox theory of motion is true, and that whenever two objects O_1 and O_2 have the same length, then the indivisible parts that compose O_1 may be put in 1-1 correspondence with the indivisible parts that compose O_2, then the argument is *sound*. So Zeno no longer looks foolish. Furthermore, this interpretation is close in spirit to traditional interpretations, and what is found in Aristotle. Additionally, it does not presuppose anything as substantial as an atomic theory of space and time.

There are also some concerns. First, there is no mention in Aristotle of the A's, B's, and C's being indivisible. But in reply, note that there is nothing in Aristotle's text that is inconsistent with this assumption either. Aristotle may have simply assumed that the reader was familiar with the assumption, or didn't feel that it was critical for his refutation, and so didn't mention it.

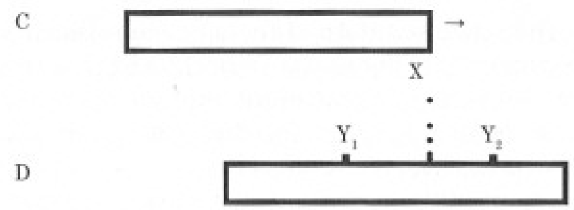
A second worry is that this interpretation of the argument is much more complicated than anything that appears in Aristotle, and that this counts against it. This concern is not without merit, but nor is it fatal. A complicated argument can have a simple refutation, and given that there is no reason to think that Aristotle took himself to be representing all the twists and turns of the Stadium argument in his brief summary in the Physics, Aristotle's quick presentation need not be taken as evidence against a complicated argument on Zeno's part. If a slightly more complicated interpretation has advantages that simpler interpretations lack, then it deserves consideration.

§4: Aristotle's Reply

How then does Aristotle's reply fare? Given this interpretation of the Stadium paradox, does Aristotle's reply miss the point, or is it a genuine objection?

Let us make some remarks about the interpretation itself, to pave the way for a discussion of Aristotle's criticism. One of the main questions the argument raises is how the passage of time should be quantified. In his argument, Zeno implicitly assumes that when one body C is moving past another body D, the passage of time can be quantified by counting the number of indivisible parts of C that are passed by some fixed part of D. There is a certain intuitive appeal in trying to quantify the passage of time in this way—after all, we often quantify the passage of time by counting the number of parts of a clock's circumference that a clock's hand passes over. What the Stadium paradox shows, however, is that when multiple bodies are moving in different ways, the various standards that arise for quantifying the passage of time in this way can clash with each other. This is how the paradox is produced.

Let us be a more explicit about what Zeno assumes. When a body C moves past a body D, the passage of time is quantified by counting the number of indivisible parts of D that are passed by some fixed part of C. So if it is assumed that a point X from body C is initially lined up with a point Y_1 from D, and after some time, the point X from body C is lined up with a point Y_2 from D, and that these two bodies are in the process of moving past each other in the usual way, then the amount of time that has passed during this motion can be measured by the number of indivisible points between Y_1 and Y_2. (See Figure 8.) Let us call this thesis about how to quantify the passage of time the "Zenonian theory of time measurement."[12]



The point X from C lines up with some point from D. At the start of the motion, the point X lines up with Y_1, and at the end of the motion it lines up with Y_2.

Figure 8

The fact that there is a 1-1 correspondence between the instants of time that pass during the motion, and the indivisible parts of the body D between Y_1 and Y_2, makes the Zenonian theory of time measurement compelling.

But even before going through the details of the Stadium argument, one might have worries about a Zenonian theory of time measurement. For according to a Zenonian theory of time measurement, the amount of time it takes the body C to complete its motion depends only on the number of indivisible points between Y_1 and Y_2, and not on the speed of C (or even the speed of D, should it also be moving). This is precisely the problem that Aristotle identifies in the passage already cited (239b33):

> The fallacy consists in requiring that a body traveling at an equal speed travels for an equal time past a moving body and a body of the same size at rest. That is false.

In raising this point, Aristotle is not simply pointing out that something that Zeno has derived is false. As discussed already, that would not be an effective criticism of an argument that proceeds via the method of *reductio ad absurdum*. Rather, what Aristotle is pointing out is that Zeno's argument presupposes a theory of time measurement, and that this theory of time measurement is clearly false. And so, Aristotle would argue, it is *incorrect* to say that Zeno's argument merely presupposes (i) that time and space consist of indivisible instants and points, (ii) the orthodox theory of motion, and (iii) whenever two objects O_1 and O_2 have the same length, then the indivisible parts that compose O_1 may be put in 1-1 correspondence with the indivisible parts that compose O_2. Aristotle is pointing out that Zeno's argument also presupposes (iv) a Zenonian theory of time measurement. This fourth presupposition is false, for the reason Aristotle suggests, and so the paradox is illusory. Seen this way, Aristotle's objection has considerable force.[13]

Still, several important clusters of questions remain open at this point. First—what is the *right* way to quantify the passage of time? Does modifying the Zenonian theory of time measurement in some trivial way get us a working alternative? It is difficult to assess the merits of Zeno's argument and Aristotle's reply until we have a sense of how closely the Zenonian theory of time measurement lies to the truth.

Second, note that it is not clear that the Stadium paradox *really does* presuppose *any* theory of time measurement. One could state the conclusion of the Stadium argument in this form:

Conclusion 1: The amount of time that passes during the motion is equal to two times the amount of time that passes during the motion.

Any attempt to argue for a conclusion of this sort must presuppose some sort of theory of time measurement, because this conclusion is about *amounts of time*.

But one does not have to state the conclusion in this form. One can state it as follows:

Conclusion 2: The number of instants of time that pass during the motion is equal to two times the number of instants of time that pass during the motion.

Conclusion 2 makes a claim only about the *number of instants* of time that pass during the motion. One might be agnostic about whether this is the same as the *amount of time* that passes during the motion. In fact, if one takes this to be the correct formulation of the conclusion of the Stadium argument, then *no* claim about *amounts* of time is ever made or assumed at any point of the Stadium paradox. But if Zeno gets his paradoxical conclusion without assuming anything about amounts of time, then it is difficult to see how Aristotle's criticism—which concerns amounts of time—can be relevant to Zeno's argument in any way.

Until these matters are more deeply understood, neither Aristotle nor Zeno can be declared victor. The resources of ancient Greek mathematics were not sufficiently rich to move the debate beyond this point; instead, we must turn to modern mathematics for help.

§5: Cantor, Borel, and the Infinite

In the late 1800s, Cantor developed his famous theory of infinite sets.[14] Central to his theory of infinite sets was the idea that two sets have the same size (or "cardinality"), just in case there is a 1-1 correspondence between them. It took some courage to develop a theory of infinite sets around this hypothesis, as there is a 1-1 correspondence between the set of natural numbers and the set of even numbers, and so, according to Cantor, the set of natural numbers has the same size as the set of even numbers, even though the even numbers form a proper subset of the set of natural numbers. Although one might worry that contradictions lurk around the corner when one says such things, Cantor developed the theory of infinite sets in such detail that it became clear that there really was nothing deeply problematic in sets having the same size as some of their proper subsets.

One theorem proved by Cantor that will be of interest to us is that *all* infinite numbers x satisfy the equation $x=2x$. Let us apply this result. Let x be the number of instants of time that pass during some particular motion of some body. Then, if x is infinite, $x=2x$; i.e., the number of instants of time that pass during the motion is equal to two times the number of instants of time that pass during the motion. But this, of course, is just the conclusion of the Stadium paradox—specifically, Conclusion 2. And so what is learned is that there is absolutely nothing paradoxical about the Stadium paradox—its conclusion is just a garden variety fact about infinite sets. The version of the Stadium argument presented in §3 is *valid*—and if it is assumed that time and space consist of indivisible instants and points, that the orthodox theory of motion is true, and that whenever two objects O_1 and O_2 have the same length, then the indivisible parts that compose O_1 may be put in 1-1 correspondence with the indivisible parts that compose O_2, then it is also *sound*.

A clarificatory remark needs to be made about atomic theories of space and time. Let x be the number of instants of time that pass during the motion of the A's, B's and C's. According to an atomic theory of space and time, x will be finite. Now, although all infinite numbers satisfy $x=2x$, no non-zero finite numbers satisfy this equation. So, given an atomic theory of space, it really *is* paradoxical to assert that the number of instants of time x that pass during the motion satisfies $x=2x$. But this is nothing strange. It is simply a reflection of the fact shown in §2, that an atomic theory of space and time, conjoined with an orthodox theory of motion, is inconsistent. The Stadium argument is still valid—though, of course, if an atomic theory of space and time is true, then the orthodox theory of motion cannot also be true, and so the Stadium argument turns out to be unsound.

But other important questions still remain unanswered at this point. Consider the set of reals (0,1) (the set of real numbers between 0 and 1), and the set of reals (2,4) (the set of real numbers between 2 and 4). Both of these sets have the same size, in the sense that there is a 1-1 correspondence between them—the function $f(x)=2(x+1)$ is such a 1-1 mapping. And yet, there is surely *some* sense in which the set of reals (2,4) is "bigger" than the set of reals (0,1), because the former spans two units, while the later spans only one unit. What this suggests is that there ought to be some *other* way of talking about the size of sets of real numbers, according to which two sets can have *different* sizes, even if there is a 1-1 correspondence between them. In other words, we ought to be able to say that some sets are bigger than other sets, even when they have the same size in Cantor's *set theoretic* sense.

It turns out that modern mathematics is quite capable of accommodating this intuition. Mathematicians do so by defining what is known as a "measure" on sets of real numbers; i.e., a function μ from sets of real numbers to positive real numbers, such that, informally speaking, μ tells us how "big" a given set of real numbers is. Given sets A and B of real numbers, say that A is bigger than B *in measure* just in case $\mu[A]<\mu[B]$. Moreover, in the case of a set of real numbers of the form (x_1, x_2), it is assumed that $\mu[(x_1, x_2)] = x_2-x_1$. So, for instance, $\mu[(0,1)]=1$, and $\mu[(2,4)]=2$, and thus the set (0,1) is smaller in measure than the set (2,4). Borel's work in 1894 was one of the first contributions to the modern theory of measure, though the work of Hausdorff and Lebesgue was of great significance as well. Although the modern theory of measure is not without problems (for instance, not all sets of real numbers can be given measures), it is a very simple way to compare the "sizes" of many sets of real numbers when cardinality is not the main concern. Let us call this conception of size a *measure theoretic* conception of size, as opposed to Cantor's *set theoretic* conception of size, which revolves around the notion of a 1-1 correspondence.[15]

Now, imagine an event E_1 happens at time t_1, and at a later time t_2, an event E_2 happens. How should the amount of time that has passed between E_1 and E_2 be quantifed? Consider the set of instants between E_1 and E_2—i.e., the set (t_1, t_2). The size of (t_1, t_2) in the set theoretic sense will be of little interest to us, as it will be the same as the sizes of (t_1, t_2+2) or (t_1-7, t_2) in the set theoretic sense. When we are interested in quantifying the amount of time that passes between events E_1 and E_2, we will want to consider the size of (t_1, t_2) in the *measure theoretic* sense. So, when measuring stretches of time, it is size in the *measure theoretic* sense, as opposed to the set theoretic sense, that matters.

Let us turn back to a Zenonian theory of time measurement. According to such a theory, when a body C moves past a body D, the amount of time that passes can be quantified by counting the number of indivisible parts of D that are passed by some fixed part of C. This amounts to measuring of the size of the set of instants that have passed in the *set theoretic* sense. But it is clear that if what is of interest is the amount of time that passes in the ordinary sense—i.e., in the sense according to which more time passes from 2pm to 4pm than passes from noon to 1pm—then a Zenonian theory of time measurement is *not* what should be considered. Instead, the sizes of sets of instants of time should be compared in the *measure theoretic* sense.

Of course, one might say that Zeno is perfectly entitled to measure the passage of time in any way he chooses—it's his argument after all. No one can prohibit Zeno from talking about the sizes of sets of instants of in the set theoretic sense. But if that is all Zeno is interested in, then his conclusion—what I have called Conclusion 2—is *not* paradoxical.

The analysis may be summarized as follows. Zeno's Stadium paradox involves the quantification of stretches of time. If Zeno is interested in quantifications of time that conform with the everyday way in which we talk about "amounts of time" passing—that is, if Zeno is interested in quantifying the passage of time in the *measure theoretic* sense—then Zeno's conclusion must be something like Conclusion 1, and Aristotle's criticism of the argument is effective (as discussed in §4.) If, on the other hand, one thinks that Zeno fully intended to quantify the passage of time in the *set-theoretic* sense, then his conclusion—in this case, Conclusion 2—is not paradoxical.

Instead of there being a real paradox hidden in the Stadium argument, there is a series of intricate lessons about time and the way in which it can be quantified. Rather than trying to view Zeno and Aristotle as villain and victor, it is more appropriate to view them as engaged in a sophisticated discussion about difficult mathematical concepts that were not systematically treated until millennia later. Neither Zeno nor Aristotle may have seen to the bottom of things, but neither of them deserve the dunce's cap either.[16]

University of Chicago

NOTES

1. J. Barnes, *The Presocratic Philosophers*, Vol. 1 (London: Routledge, 1979), pp. 236–237.

2. P. Tannery, *Pour l'Histoire de la Science Hellène* (Paris: Gauthier-Villars, 1930), pp. 247–261.

3. G. Owen, "Zeno and the Mathematicians," *Proceedings of the Aristotelian Society*, vol. 58 (1957–1958), pp. 199–222.

4. For a closer treatment, see J. Farris, *The Paradoxes of Zeno* (Brookfield, Vt.: Avebury, 1996), chap. 6.

5. We say that a body X *moves alongside* a body Y when it makes a transition from the configuration:

to the configuration:

6. R. Sorabji, *Time, Creation and the Continuum* (Ithaca, N.Y.: Cornell University Press 1983), pp. 331–332.

7. N. Booth, "Zeno's Paradoxes," *The Journal of Hellenic Studies*, vol. 77, pt. 2 (1957), pp. 187–201; see p. 188.

8. Tannery, *Pour l'Histoire de la Science Hellène*; Owen, "Zeno and the Mathematicians." See also Farris, *The Paradoxes of Zeno*, for a more comprehensive survey.

9. B. Russell gives a simple argument for a similar conclusion in *Principles of Mathematics* (New York: The Norton Library, 1964), §322, pp. 342–344.

10. Owen, "Zeno and the Mathematicians."

11. That Zeno had at least forty arguments comes to us on the authority of Proclus—see J. Dillon and G. Morrow, *Proclus' Commentary on Plato's Parmenides* (Princeton, N.J.: Princeton University Press, 1987). For a critical discussion of how seriously we should take Proclus on this, see J. Dillon, "Proclus and the Forty Logoi of Zeno," *Illinois Classical Studies*, vol. 11 (1986), pp. 35–41; and H. Tarrant, "More on Zeno's Forty Logoi," *Illinois Classical Studies*, vol. 15, no. 1 (Spring 1990), pp. 23–37.

12. Zeno himself did not have a theory of time measurement, because as a Parmenidean, he held that time was an illusion—nevertheless, insofar as this was the view about the measurement of time that, at least for the purposes of the Stadium argument, he took his opponents to have, we shall call this theory of time measurement "Zenonian."

13. Aristotle's dissatisfaction with the Stadium paradox is overdetermined. In particular, Aristotle thinks that magnitudes are not composed of indivisibles, which means that he disagrees with one of the most basic premises of the Stadium paradox as I have presented it. (Book VI, Part 1, of Aristotle's Physics, especially 231b21–232a14, contains Aristotle's discussion of why magnitudes cannot be composed of indivisibles.) In addition, however, Aristotle presents the refutation of the Stadium paradox already cited from 239b33, which does not presuppose that magnitudes are not composed of indivisibles. For the purposes of this paper, we consider only this later refutation.

14. For a modern presentation, see P. Halmos, *Naïve Set Theory* (New York: Springer Verlag, 1974).

15. For a fuller discussion of measure theory, and its connection with other paradoxes of Zeno, see B. Skyrms, "Zeno's Paradox of Measure," in *Physics, Philosophy, and Psychoanalysis: Essays in Honor of Adolf Grunbaum*, ed. R.

Cohen and L. Laudan (Dordrecht: D. Reidl, 1983), pp. 223–254; and A. Grunbaum, *Modern Science and Zeno's Paradoxes* (London: George Allen and Unwin, 1968), chap. 3.

16. Thanks to Tom Tuozzo for valuable comments and discussion on this paper.

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HUMAN NATURE AND MORAL EDUCATION IN MENCIUS, XUNZI, HOBBES, AND ROUSSEAU

Eric Schwitzgebel

Mencius, Xunzi, Hobbes, and Rousseau are all well known for their discussions of "human nature." It will be argued in this essay that, to some degree of approximation, their views on human nature can be understood as views about the proper course of moral education and that, consequently, a picture of moral development stands near the center of each man's philosophy. We can then explore empirically which philosopher was nearest the truth.

1. The "State of Nature"

The dispute between Hobbes and Rousseau regarding human nature is generally cast—and was indeed by Rousseau himself sometimes cast—as a dispute about what people (or "man") would be like in the "state of nature," a state without social structures or government. Hobbes famously writes in the *Leviathan* that the "naturall condition of mankind"—his condition prior to establishment of the state—is one of misery and "Warre, where every man is Enemy to every man" and life is "solitary, poore, nasty, brutish, and short."[1] We are propelled into violent competition by the desire for limited goods and for glory, and due to our relative indifference to the suffering of others. When a man in the state of nature sees something he wants—such as the goods or wife of another man—he will try to obtain it, if he can do so consistently with his own safety, regardless of the pain or death it may bring to others. The result is continual insecurity and strife, and the failure of any stable agriculture or industry, until men are eventually persuaded to submit themselves to a government for their own protection.

Rousseau, equally famously, paints a very different picture of the "state of nature" in his *Discourse on the Origin of Inequality*. Man in the